

THE DIFFEOMORPHISM TYPE OF SMALL HYPERPLANE ARRANGEMENTS IS COMBINATORIALLY DETERMINED

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ABSTRACT. It is known that there exist hyperplane arrangements with same underlying matroid that admit non-homotopy equivalent complement manifolds. In this work we show that, in any rank, complex central hyperplane arrangements with up to 7 hyperplanes and same underlying matroid are isotopic. In particular, the diffeomorphism type of the complement manifold and the Milnor fiber and fibration of these arrangements are combinatorially determined, that is, they depend uniquely on the underlying matroid. To do this, we associate to every such matroid a topological space, that we call the *reduced realization space*; its connectedness — showed by means of symbolic computation — implies the desired result.

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INTRODUCTION

The central problem in hyperplane arrangement theory is to determine whether the topology or the homotopy type of the complement manifold of an arrangement is described by the combinatorial properties of the arrangement itself. This theory was first developed in [Arn69] with motivations in the study of configuration spaces.

One of the seminal works on the homotopy theory of complex hyperplane arrangements is the computation of the integer cohomology algebra structure of the complement manifold of an arrangement by Orlik and Solomon [OS80]. Motivated by work of Arnol'd, they exploited techniques of Brieskorn [Bri73] to provide a presentation of this cohomology algebra in terms of generators and relations that depends uniquely on the underlying matroid of the arrangement.

The result of [OS80] has generated a lot of new conjectures and problems, asking which homotopy invariants of the complement manifold of an arrangement are combinatorially determined. A cornerstone in this direction is the isotopy theorem proved by Randell in [Ran89]. It states that the diffeomorphism type of the complement manifold does not change through an isotopy, that is a smooth one-parameter family of arrangements with constant underlying matroid. Afterwards, in [Ran97] Randell showed similar results for more sophisticated invariants such as the Milnor fiber and fibration of an arrangement (compare Definition 1.3).

Randell’s isotopy theorem can be actually reformulated in terms of matroid realization spaces, that are related to the well studied matroid stratification of the Grassmannian. In their celebrated paper [GGMS87], Gel’fand, Goresky, MacPherson and Serganova studied this stratification and described some of its equivalent reformulations. In particular, Randell’s results give rise to the problem of describing the connected components of the matroid strata of the Grassmannian.

On the other hand, in [Ryb11] Rybnikov found an example of arrangements with same underlying matroid but non-isomorphic fundamental groups of the corresponding complement manifolds. However, in many remarkable cases the topology of the complement manifold can be still recovered simply by the combinatorial data. Thus, one important problem is to characterize wider families of arrangements for which Randell’s isotopy theorem holds.

Several results in this direction appeared in the literature. In particular, Jiang and Yau [JY98], Nazir and Yoshinaga [NY12] and Amram, Teicher and Ye [ATY13] focused their attention on some specific classes of line arrangements in the complex projective plane. However, their techniques seem hardly generalizable to higher dimensions.

To every matroid M we can associate the set of hyperplane arrangements having M as underlying matroid. Such set has a natural topological structure as a subset of a space of matrices, and it is called the *realization space* of M . Here, building on previous results of Delucchi and the second-named author (see [DS15]), we associate to a matroid another topological space, called its *reduced realization space* (Definition 2.2). As the name suggests, the latter is a subset of the realization space, and it is obtained by considering hyperplane arrangements of a given shape. Such shape is determined by what we call the *normal frame* of a matrix (Definition 2.1). Exploiting some ideas from [BL73] and [Rui13] we study it, and finally we describe (Proposition 2.1) how the connectedness of the reduced realization space is related to the one of the “classic” realization space. Moreover, we show by means of symbolic computation and elementary algebraic geometry arguments that for any matroid with ground set of up to 7 elements the associated reduced realization space is either empty or connected.

So that, from the results of [Ran89] and [Ran97] we can conclude that the diffeomorphism type of the complement manifold and the Milnor fiber and fibrations of complex central hyperplane arrangements with up to 7 hyperplane are combinatorially determined, that is, they depend uniquely on the underlying matroid of these arrangements.

Overview. Section 1 contains some basic definitions on matroids and complex hyperplane arrangements. In Section 2, we introduce the normal frame of a matrix and the reduced realization space of a matroid, and we deduce some of their properties. Section 3 is devoted to applications in the study of the isotopy type of arrangements with up to 7 hyperplanes. For readability’s sake we postpone some of the technical computations to Appendix A.

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1. MATROIDS AND ARRANGEMENTS

In this section we provide a quick review of some basic definitions and results about matroids and arrangements. We refer to the book [Oxl92] for a detailed treatment of matroid theory and we point to [OT92] for a general theory of arrangements and to [FR00] for a survey of their homotopy theory.

1.1. Matroids. A *matroid* M is a pair (E, \mathfrak{I}) , where E is a finite *ground set* and $\mathfrak{I} \subseteq 2^E$ is a family of subsets of E satisfying the following three conditions:

- (I1) $\emptyset \in \mathfrak{I}$;
- (I2) if $I \in \mathfrak{I}$ and $J \subseteq I$, then $J \in \mathfrak{I}$;
- (I3) if I and J are in \mathfrak{I} and $|I| < |J|$, then there exists an element $e \in J \setminus I$ such that $I \cup \{e\} \in \mathfrak{I}$.

The elements of \mathfrak{I} are called the *independent sets* of M . Maximal independent sets (with respect to inclusion) are called *bases*, and the set of bases of M will be denoted by \mathfrak{B} . By definition, the *rank* of a subset $S \subseteq E$ is

$$\text{rk}(S) = \max\{|S \cap B| \mid B \in \mathfrak{B}\}$$

and the *rank* of the matroid M is the rank of the ground set E .

A rank d matroid M with ground set $E = \{1, \dots, m\}$ is called *realizable* over \mathbb{C} if there exists a matrix $A \in M_{d,m}(\mathbb{C})$ of d rows and m columns with complex coefficients such that

$$\left\{ J \subseteq E \mid \{A^j\}_{j \in J} \text{ is linearly independent over } \mathbb{C} \right\}$$

is the family of independent sets of M . Here, A^j denotes the j -th column of A . We say that A *realizes* M over \mathbb{C} .

Definition 1.1 (Realization space). For a rank d matroid M with ground set $E = \{1, \dots, m\}$ the *realization space* of M over \mathbb{C} is the set $\mathcal{R}_{\mathbb{C}}(M)$ of matrices $A \in M_{d,m}(\mathbb{C})$ that satisfy the following condition:

- A realizes M over \mathbb{C} .

We endow $\mathcal{R}_{\mathbb{C}}(M)$ with the subspace topology of $M_{d,m}(\mathbb{C})$.

If $\mathcal{R}_{\mathbb{C}}(M)$ is empty, that is, there are no matrices that realize M over \mathbb{C} , we say that M is *non-realizable* over \mathbb{C} .

1.2. Arrangements. Any finite collection $\mathcal{A} = \{H_1, \dots, H_m\}$ of affine subspaces in \mathbb{C}^d will be called an *arrangement*. Its *complement manifold* $M(\mathcal{A})$ is the complement of the union of the H_i in \mathbb{C}^d . The arrangement is *central* if every H_i contains the origin. For an arrangement $\mathcal{A} = \{H_1, \dots, H_m\}$ in \mathbb{C}^d we assign a *rank* to each subset $S \subseteq \{1, \dots, m\}$ by setting

$$\text{rk}_{\mathcal{A}}(S) = \text{codim} \bigcap_{i \in S} H_i$$

(where we define the empty set to have codimension $d + 1$).

We say that the arrangements $\mathcal{A} = \{H_1, \dots, H_m\}$ and $\mathcal{B} = \{K_1, \dots, K_m\}$ have the same *combinatorial type* if the functions $\text{rk}_{\mathcal{A}}$ and $\text{rk}_{\mathcal{B}}$ coincide.

Given an open interval $(a, b) \subseteq \mathbb{R}$, a *smooth one-parameter family* of arrangements is a collection $\{\mathcal{A}_t\}_{t \in (a,b)}$ of arrangements $\mathcal{A}_t = \{H_1(t), \dots, H_m(t)\}$ in \mathbb{C}^d such that there exist smooth functions from (a, b) to \mathbb{C} for the coefficients of defining equations of the subspaces $H_i(t)$. With a slight abuse of notation we write \mathcal{A}_t for $\{\mathcal{A}_t\}_{t \in (a,b)}$, omitting the interval of parameters (a, b) .

Definition 1.2 (Isotopic arrangements). A smooth one-parameter family of arrangements \mathcal{A}_t is an *isotopy* if for any t_1 and t_2 the arrangements \mathcal{A}_{t_1} and \mathcal{A}_{t_2} have the same combinatorial type. In this case we say that \mathcal{A}_{t_1} and \mathcal{A}_{t_2} are *isotopic*.

The following theorem, sometimes referred to as “isotopy theorem” was proved by Randell [Ran89]. This is one of the pillars on which our work is based, allowing us, from now on, to focus on isotopic arrangements.

Theorem 1.1 ([Ran89]). *If \mathcal{A}_{t_1} and \mathcal{A}_{t_2} are isotopic arrangements, then the complement manifolds $M(\mathcal{A}_{t_1})$ and $M(\mathcal{A}_{t_2})$ are diffeomorphic.*

A *hyperplane arrangement* is an arrangement of codimension 1 subspaces. Again, a hyperplane arrangement is *central* if each of its subspaces is linear. For a central hyperplane arrangement $\mathcal{A} = \{H_1, \dots, H_m\}$ in \mathbb{C}^d pick linear forms α_i in the dual space $(\mathbb{C}^d)^*$ with $H_i = \ker \alpha_i$. The *underlying matroid* of \mathcal{A} is by definition the matroid $M_{\mathcal{A}}$ with ground set $E_{\mathcal{A}} = \{1, \dots, m\}$ and

$$\mathfrak{I}_{\mathcal{A}} = \{S \subseteq E \mid \{\alpha_i\}_{i \in S} \text{ is linearly independent over } \mathbb{C}\}$$

as independent sets. Clearly, the matroid $M_{\mathcal{A}}$ does not depend on the choice of the linear forms α_i . The *rank* of \mathcal{A} is by definition the rank of $M_{\mathcal{A}}$.

Notice that a smooth one-parameter family \mathcal{A}_t of central hyperplane arrangements is an isotopy if and only if $M_{\mathcal{A}_{t_1}} = M_{\mathcal{A}_{t_2}}$ for any t_1 and t_2 .

Definition 1.3 (Milnor fiber and fibration). Given linear forms $\alpha_i \in (\mathbb{C}^d)^*$ with $H_i = \ker \alpha_i$, the polynomial $Q_{\mathcal{A}} = \prod_{i=1}^m \alpha_i$ is homogeneous of degree m and can be considered as a map

$$Q_{\mathcal{A}} : M(\mathcal{A}) \longrightarrow \mathbb{C}^*$$

that is the projection of a fiber bundle called the *Milnor fibration* of the arrangement (see [Mil68]). The *Milnor fiber* is then the fiber $F_{\mathcal{A}} = Q_{\mathcal{A}}^{-1}(1)$.

The subsequent theorem proved by Randell in [Ran97] states that the Milnor fiber and fibration are also invariants for isotopic arrangements.

Theorem 1.2 ([Ran97]). *Let \mathcal{A}_t be a smooth one-parameter family of central hyperplane arrangements. If \mathcal{A}_t is an isotopy, then for any t_1 and t_2 the Milnor fibrations $Q_{\mathcal{A}_{t_1}}$ and $Q_{\mathcal{A}_{t_2}}$ are isomorphic fiber bundles.*

2. REDUCED REALIZATION SPACES

Throughout this section we suppose that, given a rank d matroid M with ground set $E = \{1, \dots, m\}$, the set $\{1, \dots, d\}$ is a basis of M . We can always assume this after relabelling the elements of the ground set.

Our goal is to introduce a subspace $\mathcal{R}_{\mathbb{C}}^R(M)$ of the realization space $\mathcal{R}_{\mathbb{C}}(M)$ that contains information about the realizability of M over \mathbb{C} and the connectedness of $\mathcal{R}_{\mathbb{C}}(M)$, but it is easier to describe than the full space $\mathcal{R}_{\mathbb{C}}(M)$.

Suppose in fact that $A \in M_{d,m}(\mathbb{C})$ realizes M over \mathbb{C} , namely $A \in \mathcal{R}_{\mathbb{C}}(M)$. Since we assume that $\{1, \dots, d\}$ is a basis for M , we can perform a change of coordinates in \mathbb{C}^d in such a way that the columns A^1, \dots, A^d of A become the standard basis. The new matrix we obtain realizes M over \mathbb{C} as well. At this point, we can multiply every row of A by a non-zero scalar without modifying the realizability property. Therefore, for a matrix $A \in M_{d,m}(\mathbb{C})$ realizing M over \mathbb{C} we can try to find an invertible matrix $G \in GL_d(\mathbb{C})$ of rank d and a complex non-singular diagonal matrix D of rank m so that GAD has as many zeros and ones as possible, and still realizes M over \mathbb{C} . Our new space will correspond to the set of these “reduced” matrices.

In order to define precisely and to be able to manipulate the object we are going to define, we need a somehow technical notion, the *normal frame* of a matrix, which we describe here. Let us consider a matrix $Q \in M_{n,r}(\mathbb{C})$ of n rows and r columns with complex coefficients and let us associate to Q a board $S_0(Q)$ of n rows and r columns with black squares in correspondence to the zero entries of Q and white

squares in correspondence to the non-zero entries of Q . At this point, we perform the following sequence of operations on the board $S_0(Q)$:

- (O1) For each column of $S_0(Q)$ we color blue the first white square from the top to the bottom. We call this board $S_1(Q)$;
- (O2) For each row of $S_1(Q)$ we color red the first white square from the left to the right. We call this board $S_2(Q)$;
- (O3) We color green each blue or red square of $S_2(Q)$. We call this board $S(Q)$.

Definition 2.1 (Normal frame). The *normal frame* of a matrix $Q \in M_{n,r}(\mathbb{C})$ is

$$\mathcal{P}_Q = \{(i, j) \in \{1, \dots, n\} \times \{1, \dots, r\} \mid \text{the } (i, j)\text{-th square of } S(Q) \text{ is green}\}$$

We are now ready to define the reduced realization space of a matroid.

Definition 2.2 (Reduced realization space). For a rank d matroid M with ground set $E = \{1, \dots, m\}$ and $\{1, \dots, d\}$ as basis, the *reduced realization space* of M over \mathbb{C} is the set $\mathcal{R}_{\mathbb{C}}^R(M)$ of matrices $A \in M_{d,m}(\mathbb{C})$ that satisfy the following conditions:

- (C1) A realizes M over \mathbb{C} , that is, A belongs to $\mathcal{R}_{\mathbb{C}}(M)$;
- (C2) A is of the form $(I_d | \tilde{A})$, where I_d is the $d \times d$ identity matrix;
- (C3) the entries of \tilde{A} that are in the normal frame $\mathcal{P}_{\tilde{A}}$ equal 1.

We endow $\mathcal{R}_{\mathbb{C}}^R(M)$ with the subspace topology of $M_{d,m}(\mathbb{C})$.

Note 2.1. For a matrix $A \in M_{d,m}(\mathbb{C})$, condition (C1) is equivalent to

$$(*) \quad \begin{cases} \det(A^{j_1} | \dots | A^{j_d}) \neq 0 & \text{if } \{j_1, \dots, j_d\} \in \mathfrak{B} \\ \det(A^{j_1} | \dots | A^{j_d}) = 0 & \text{if } \{j_1, \dots, j_d\} \notin \mathfrak{B} \end{cases}$$

where A^j denotes the j -th column of A and \mathfrak{B} is the set of bases of M . For $1 \leq i \leq d$ and $d+1 \leq j \leq m$, if we consider the d -uples

$$(\{1, \dots, d\} \setminus \{i\}) \cup \{j\}$$

it follows from $(*)$ that, given matrices \tilde{A}_1 and \tilde{A}_2 in $M_{d,m-d}(\mathbb{C})$ with $(I_d | \tilde{A}_1)$ and $(I_d | \tilde{A}_2)$ in $\mathcal{R}_{\mathbb{C}}^R(M)$, the board $S_0(\tilde{A}_1)$ equals $S_0(\tilde{A}_2)$. Hence, all matrices \tilde{A} in $M_{d,m-d}(\mathbb{C})$ with $(I_d | \tilde{A})$ in $\mathcal{R}_{\mathbb{C}}^R(M)$ have the same normal frame. This, together with condition (C2) and $(*)$, implies that $\mathcal{R}_{\mathbb{C}}^R(M)$ can be written as a subset of $M_{d,m}(\mathbb{C})$ satisfying a system of equalities and inequalities of polynomial type.

The subsequent proposition will clarify how the spaces $\mathcal{R}_{\mathbb{C}}(M)$ and $\mathcal{R}_{\mathbb{C}}^R(M)$ are related. In particular, it shows that the connectedness of $\mathcal{R}_{\mathbb{C}}^R(M)$ implies the one of $\mathcal{R}_{\mathbb{C}}(M)$. This fact is a direct consequence of the connectedness of the complex linear group and of the complex torus.

Proposition 2.1. *For a rank d matroid M with ground set $E = \{1, \dots, m\}$ and $\{1, \dots, d\}$ as basis, let $A \in \mathcal{R}_{\mathbb{C}}(M)$ be a matrix that realizes M over \mathbb{C} . Then, there exist an invertible matrix $G \in GL_d(\mathbb{C})$ of rank d and a complex non-singular diagonal matrix D of rank m such that $GAD \in \mathcal{R}_{\mathbb{C}}(M)$. In particular, the following properties hold:*

- (P1) $\mathcal{R}_{\mathbb{C}}(M) \neq \emptyset$ if and only if $\mathcal{R}_{\mathbb{C}}^R(M) \neq \emptyset$;
- (P2) If $\mathcal{R}_{\mathbb{C}}^R(M)$ is connected, so is $\mathcal{R}_{\mathbb{C}}(M)$.

To show this result we need at first to prove some technical lemmas.

Lemma 2.1. *For a matrix $Q \in M_{n,r}(\mathbb{C})$ with at least a non-zero entry, consider the board $S(Q)$ associated to Q . Then, there exists a line (row or column) of $S(Q)$ that contains exactly one green square and such that the board obtained from $S(Q)$ by deleting this line coincides with the one obtained from $S_0(Q)$ by deleting such line and then performing the steps (O1), (O2) and (O3).*

Proof. Without loss of generality we can assume that each line (row or column) of $S_0(Q)$ contains at least a white square. Otherwise, it suffices to delete that black line and study the problem for a smaller board. Set

$$\nu = \max \{i \in \{1, \dots, n\} \mid \text{the } i\text{-th row of } S_1(Q) \text{ contains a blue square}\}$$

and notice that under the assumption that each line of $S_0(Q)$ contains at least a white square, this number ν is well defined. We distinguish two cases:

- If $1 \leq \nu < n$, then the statement follows by considering the $(\nu + 1)$ -th row.
- If $\nu = n$, then it suffices to consider the first column for which this maximum is attained. \square

Lemma 2.2. *Given a matrix $Q \in M_{n,r}(\mathbb{C})$ there exist complex non-singular diagonal matrices D_1 of rank n and D_2 of rank r such that the entries of $D_1 Q D_2$ that belong to the normal frame \mathcal{P}_Q of Q equal 1.*

Proof. Our proof exploits the same ideas of [BL73, Proposition 2.7]. We proceed by induction on the cardinality of the normal frame \mathcal{P}_Q of Q . If $|\mathcal{P}_Q| = 0$ there is nothing to prove, since this condition is equivalent to the fact that all entries of Q are zero. Now, let us assume our statement true for all matrices with normal frame of cardinality strictly less than k and let us consider a matrix Q with normal frame \mathcal{P}_Q of k elements. From Lemma 2.1 we know that there exists a line (row or column) of the board $S(Q)$ that contains exactly one green square and such that the board obtained from $S(Q)$ by deleting this line coincides with the one obtained from $S_0(Q)$ and then performing the steps (O1), (O2) and (O3). Notice that our proof will be essentially the same if we suppose that line is a column. Hence, let us assume that line is the i -th row of $S(Q)$. Let us denote by (i, j) the position of the unique green square placed in it. In particular, the entry q_{ij} of Q is non-zero. Otherwise, by definition of the algorithmic steps (O1), (O2) and (O3) there will be a black square in the position (i, j) of the board $S(Q)$. Let us denote by $\tilde{Q} \in M_{n-1,r}(\mathbb{C})$ the matrix obtained from Q by deleting its i -th row. With the second part of the statement of Lemma 2.1 we can deduce that the normal frame $\mathcal{P}_{\tilde{Q}}$ of \tilde{Q} has $k - 1$ elements. Thus, by inductive hypothesis there exist complex non-singular diagonal matrices \tilde{D}_1 of rank $n - 1$ and \tilde{D}_2 of rank r such that the entries of $\tilde{D}_1 \tilde{Q} \tilde{D}_2$ that belong to the normal frame $\mathcal{P}_{\tilde{Q}}$ of \tilde{Q} equal 1. So finally, if we define

$$D_1 = \text{diag} \left(\tilde{D}_1(1), \dots, \tilde{D}_1(i-1), \left(\tilde{D}_2(j) q_{ij} \right)^{-1}, \tilde{D}_1(i), \dots, \tilde{D}_1(n-1) \right)$$

and set $D_2 = \tilde{D}_2$, one can easily check that all entries of $D_1 Q D_2$ that belong to the normal frame \mathcal{P}_Q of Q equal 1. \square

Proof of Proposition 2.1. Let $A \in \mathcal{R}_{\mathbb{C}}(M)$ be a matrix that realizes M over \mathbb{C} . Since $\{1, \dots, d\}$ is a basis of M , there exists an invertible matrix $B \in GL_d(\mathbb{C})$ of rank d such that $BA = (I_d | Q)$, where $Q \in M_{d,m-d}(\mathbb{C})$. By Lemma 2.2 there exist complex non-singular diagonal matrices D_1 of rank d and D_2 of rank $m - d$ such that the entries of $D_1 Q D_2$ belonging to the normal frame \mathcal{P}_Q of Q equal 1. Now, set

$$D = \text{diag} (D_1(1)^{-1}, \dots, D_1(d)^{-1}, D_2(1), \dots, D_2(m-d))$$

and $G = D_1 B$. With elementary linear algebra arguments it is not hard to see that the matrix GAD realizes the matroid M over \mathbb{C} as well. Hence, condition (C1) in Definition 2.2 is satisfied. By construction the matrix GAD is of the form $(I_d | D_1 Q D_2)$, so that conditions (C2) and (C3) in Definition 2.2 are fulfilled, too.

Hence, it remains to check that properties (P1) and (P2) hold.

- Property (P1) follows directly from the first part of our statement and the set inclusion $\mathcal{R}_{\mathbb{C}}^R(M) \subseteq \mathcal{R}_{\mathbb{C}}(M)$.
- To prove that (P2) is satisfied, let us assume $\mathcal{R}_{\mathbb{C}}^R(M)$ connected. We show that under this assumption $\mathcal{R}_{\mathbb{C}}(M)$ is actually a path connected space. Since $\mathcal{R}_{\mathbb{C}}^R(M)$ can be expressed as a subset of $M_{d,m}(\mathbb{C})$ satisfying a system of polynomial equalities and inequalities (see Note 2.1), the connectedness hypothesis of $\mathcal{R}_{\mathbb{C}}^R(M)$ implies that $\mathcal{R}_{\mathbb{C}}^R(M)$ is path connected. Let A and B be matrices of $\mathcal{R}_{\mathbb{C}}(M)$. Using the first part of our statement, let G_1 and $G_2 \in GL_d(\mathbb{C})$ be invertible matrices of rank d and let D_1, D_2 be complex non-singular diagonal matrices of rank m such that G_1AD_1 and G_2BD_2 belong to $\mathcal{R}_{\mathbb{C}}^R(M)$. Since $\mathcal{R}_{\mathbb{C}}^R(M)$ is path connected, we can find a continuous path $\gamma: [0, 1] \rightarrow \mathcal{R}_{\mathbb{C}}^R(M)$ such that $\gamma(0)$ equals G_1AD_1 and $\gamma(1)$ equals G_2BD_2 . Moreover, from the inclusion $\mathcal{R}_{\mathbb{C}}^R(M) \subseteq \mathcal{R}_{\mathbb{C}}(M)$ and the fact that both these spaces are endowed with the subspace topology of $M_{d,m}(\mathbb{C})$, we see that γ is indeed a continuous path with values in $\mathcal{R}_{\mathbb{C}}(M)$. The complex linear group $GL_d(\mathbb{C})$ is path connected, since it is the complement of the complex algebraic curve $\{\det(X) = 0\}$ in $M_{d,d}(\mathbb{C})$. Also the space $D_m(\mathbb{C})$ of complex non-singular diagonal matrices of rank m is path connected, since it can be diffeomorphically identified with the complex torus $(\mathbb{C}^*)^m$. Thus, there exist continuous paths

$$\sigma_1, \sigma_2: [0, 1] \rightarrow GL_d(\mathbb{C}) \quad \text{and} \quad \tau_1, \tau_2: [0, 1] \rightarrow D_m(\mathbb{C})$$

with

$$\begin{aligned} \sigma_1(0) &= I_d, & \sigma_1(1) &= G_1, & \sigma_2(0) &= I_d, & \sigma_2(1) &= G_2, \\ \tau_1(0) &= I_m, & \tau_1(1) &= D_1, & \tau_2(0) &= I_m, & \tau_2(1) &= D_2. \end{aligned}$$

Now, consider $\Gamma_A(t) = \sigma_1(t)A\tau_1(t)$ and $\Gamma_B(t) = \sigma_2(t)B\tau_2(t)$. Again, using elementary linear algebra arguments, we can easily see that for $t \in [0, 1]$ the matrices $\Gamma_A(t)$ and $\Gamma_B(t)$ belong to $\mathcal{R}_{\mathbb{C}}(M)$. So finally, if we consider the joined path

$$\sigma(t) = \begin{cases} \Gamma_A(3t) & \text{if } t \in [0, 1/3] \\ \gamma(3t - 1) & \text{if } t \in [1/3, 2/3] \\ \Gamma_B(3 - 3t) & \text{if } t \in [2/3, 1] \end{cases}$$

we obtain a continuous path $\sigma: [0, 1] \rightarrow \mathcal{R}_{\mathbb{C}}(M)$ with $\sigma(0) = A$ and $\sigma(1) = B$. \square

3. APPLICATIONS

The aim of this section is to prove that complex central hyperplane arrangements with up to 7 hyperplanes and same underlying matroid are isotopic, improving the results of [NY12] to any rank. The central idea of our proof is to exploit the connectedness of the reduced realization space of the underlying matroid of these arrangements to apply Proposition 2.1.

Theorem 3.1. *Let $\mathcal{A} = \{H_1, \dots, H_m\}$ and $\mathcal{B} = \{K_1, \dots, K_m\}$ be rank d central hyperplane arrangements in \mathbb{C}^d with same underlying matroid. If $1 \leq d \leq m \leq 7$, then \mathcal{A} and \mathcal{B} are isotopic arrangements.*

This result implies that the diffeomorphism type of the complement manifold and the Milnor fiber and fibration of these arrangements are uniquely determined by their underlying matroid.

Corollary 3.1. *Let $\mathcal{A} = \{H_1, \dots, H_m\}$ and $\mathcal{B} = \{K_1, \dots, K_m\}$ be rank d central hyperplane arrangements in \mathbb{C}^d with same underlying matroid. If $1 \leq d \leq m \leq 7$, then the following properties are fulfilled:*

- (1) The complement manifolds $M(\mathcal{A})$ and $M(\mathcal{B})$ are diffeomorphic;
- (2) The Milnor fibrations $Q_{\mathcal{A}}$ and $Q_{\mathcal{B}}$ are isomorphic fiber bundles.

Proof. (1) follows from Theorem 1.1 and (2) is a consequence of Theorem 1.2. \square

To prove Theorem 3.1 some preliminary results are required.

Lemma 3.1. *For a rank d matroid M with ground set $E = \{1, \dots, m\}$ let A and B be two matrices that realize M over \mathbb{C} and belong to the same connected component of $\mathcal{R}_{\mathbb{C}}(M)$. Then, there exists $\epsilon > 0$ and a smooth path $\sigma: (-\epsilon, 1 + \epsilon) \rightarrow M_{d,m}(\mathbb{C})$ such that $\sigma(0) = A$, $\sigma(1) = B$ and $\sigma(t) \in \mathcal{R}_{\mathbb{C}}(M)$ for $t \in (-\epsilon, 1 + \epsilon)$.*

Proof. Let \mathfrak{B} be the set of bases of M and write

$$\mathcal{R}_{\mathbb{C}}(M) = \left\{ A \in M_{d,m}(\mathbb{C}) \mid \begin{array}{ll} \det(A^{j_1} \dots A^{j_d}) \neq 0 & \text{if } \{j_1, \dots, j_d\} \in \mathfrak{B} \\ \det(A^{j_1} \dots A^{j_d}) = 0 & \text{if } \{j_1, \dots, j_d\} \notin \mathfrak{B} \end{array} \right\}$$

where A^j denotes the j -th column of A . Hence, the space $\mathcal{R}_{\mathbb{C}}(M)$ can be expressed as a subset of $M_{d,m}(\mathbb{C})$ satisfying a system of equalities and inequalities of polynomial type. As a consequence of this, each connected component \mathcal{C} of $\mathcal{R}_{\mathbb{C}}(M)$ is actually a piecewise linear path connected space. Thus, the following property is fulfilled:

if $A, B \in \mathcal{C}$, then there exists $\epsilon > 0$ and a piecewise linear path
 $\gamma: (-\epsilon, 1 + \epsilon) \rightarrow \mathcal{C}$ with $\gamma(0) = A$ and $\gamma(1) = B$.

Since the equalities and inequalities that define $\mathcal{R}_{\mathbb{C}}(M)$ are of polynomial type, it is not hard to see that it is possible to reparametrize the path γ to find a smooth path $\sigma: (-\epsilon, 1 + \epsilon) \rightarrow M_{d,m}(\mathbb{C})$ such that $\sigma(0) = A$, $\sigma(1) = B$ and $\sigma(t) \in \mathcal{C}$ for $t \in (-\epsilon, 1 + \epsilon)$. \square

Lemma 3.2. *For a rank d matroid M with ground set $E = \{1, \dots, m\}$ and $\{1, \dots, d\}$ as basis, if $1 \leq d \leq m \leq 7$ and M is realizable over \mathbb{C} , then the space $\mathcal{R}_{\mathbb{C}}^R(M)$ is non-empty and connected.*

Proof. Since by hypothesis M is realizable over \mathbb{C} , we have that $\mathcal{R}_{\mathbb{C}}(M)$ is non-empty, and so by Proposition 2.1 we get $\mathcal{R}_{\mathbb{C}}^R(M) \neq \emptyset$. The space $\mathcal{R}_{\mathbb{C}}^R(M)$ can be expressed as a subset of $M_{d,m}(\mathbb{C})$ satisfying a system of polynomial equalities and inequalities (see Note 2.1). From [Sha13, Chapter 7, Theorem 7.1] to prove connectedness it is enough to show that $\mathcal{R}_{\mathbb{C}}^R(M)$ is irreducible in the Zariski topology. We checked this for all matroids M satisfying the hypothesis by a direct computation with the aid of the computer algebra system Sage [Dev15] (for further details, see Appendix A). \square

Proof of Theorem 3.1. Let M be the underlying matroid of the arrangements \mathcal{A} and \mathcal{B} . Up to relabelling the hyperplanes of \mathcal{A} and \mathcal{B} , let us suppose that $\{1, \dots, d\}$ is a basis of M . Pick linear forms α_i and β_i such that $H_i = \ker \alpha_i$ and $K_i = \ker \beta_i$. Let us denote by α_i^j and β_i^j the j -th component of α_i and β_i , respectively. Set $A = (\alpha_i^j)^t$ and $B = (\beta_i^j)^t$. Now, consider the space $\mathcal{R}_{\mathbb{C}}(M)$. The matrices A and B belong to $\mathcal{R}_{\mathbb{C}}(M)$. Hence, to prove that \mathcal{A} and \mathcal{B} are isotopic arrangements (compare Definition 1.2) it is enough to show that there exists $\epsilon > 0$ and a smooth path $\sigma: (-\epsilon, 1 + \epsilon) \rightarrow M_{d,m}(\mathbb{C})$ with $\sigma(0) = A$, $\sigma(1) = B$ and $\sigma(t) \in \mathcal{R}_{\mathbb{C}}(M)$ for t in $(-\epsilon, 1 + \epsilon)$. Thus, with Lemma 3.1 it suffices to check that $\mathcal{R}_{\mathbb{C}}(M)$ is connected. To see this, thanks to Proposition 2.1, we can just verify the connectedness of $\mathcal{R}_{\mathbb{C}}^R(M)$. So that, the statement follows from Lemma 3.2. \square

APPENDIX A. CHECKING CONNECTEDNESS OF REDUCED REALIZATION SPACES

We are going to show by a direct test that Lemma 3.2 holds. For a rank d matroid M with ground set $E = \{1, \dots, m\}$ and $\{1, \dots, d\}$ as basis, let us consider a matrix $G_{0,M} \in M_{d,m}(\mathbb{C})$ with all entries equal to -1 and let us perform the following sequence of operations:

- (S1) We insert a $d \times d$ identity matrix in correspondence of the first d columns of $G_{0,M}$. We call this matrix $G_{1,M}$;
- (S2) For $1 \leq i \leq d$ and $d+1 \leq j \leq m$ we set the (i, j) -th entry of $G_{1,M}$ equal 0 if $(\{1, \dots, d\} \setminus \{i\}) \cup \{j\}$ is not a basis of M . We call this matrix $G_{2,M}$;
- (S3) Let $\tilde{G}_{2,M}$ be the $d \times (m-d)$ matrix such that $G_{2,M} = (I_d | \tilde{G}_{2,M})$. We set the entries of $\tilde{G}_{2,M}$ that are in the normal frame $\mathcal{P}_{\tilde{G}_{2,M}}$ equal 1. We call this matrix $\tilde{G}_{3,M}$ and we set $G_{3,M} = (I_d | \tilde{G}_{3,M})$;
- (S4) We call s_M the number of -1 entries of $G_{3,M}$;
- (S5) We replace the -1 entries of $G_{3,M}$ with symbolic variables t_1, \dots, t_{s_M} and we call this matrix G_M .

Algorithm 1 TestIrreducibility

Require: `case` = (d, m) a pair from Equation (**).

Ensure: `True` if the reduced realization spaces of all realizable matroids of type `case` are irreducible, `False` otherwise.

- 1: **Compute** the list `subsets` of all subsets of d elements of $\{1, \dots, m\}$ and order it w.r.t. the reverse lexicographic term order.
 - 2: **for** `matroid` in `all_matroids[case]` **do**
 - 3: **Compute** the first basis for `matroid` in the list `subsets` and call it `basis`.
 - 4: **Set** $G = \text{FillMatrix}(\text{case}, \text{basis})$.
 - 5: \triangleright Computing the (in)equalities for X_{matroid} .
 - 6: **Substitute** the -1 entries of G with symbolic variables.
 - 7: **Set** `equalities` = `emptylist` and `inequalities` = `emptylist`.
 - 8: **for** `subset` in `subsets` **do**
 - 9: **Set** `det` to be the $d \times d$ minor corresponding to the submatrix of G whose columns are prescribed by `subset`.
 - 10: **if** `subset` is a basis for `matroid` **then** **Add** `det` to `inequalities`.
 - 11: **else** **Add** `det` to `equalities`.
 - 12: **end if**
 - 13: **end for**
 - 14: \triangleright Checking irreducibility of the zero set determined by only the equalities.
 - 15: **Set** `ideal` to be the ideal generated by `equalities`.
 - 16: **if** the zero set of `ideal` is not geometrically irreducible **then**
 - 17: **return** `False`.
 - 18: **end if**
 - 19: **end for**
 - 20: **return** `True`.
-

Definition A.1. For a rank d matroid M with ground set $E = \{1, \dots, m\}$ and $\{1, \dots, d\}$ as basis the *reduced variety* of M over \mathbb{C} is the quasi-projective variety X_M defined by

$$X_M = \left\{ (z_1, \dots, z_{s_M}) \in \mathbb{A}_{\mathbb{C}}^{s_M} \left| \begin{array}{ll} \det \left(G_M^{j_1} | \dots | G_M^{j_d} \right) \neq 0 & \text{if } \{j_1, \dots, j_d\} \in \mathfrak{B} \\ \det \left(G_M^{j_1} | \dots | G_M^{j_d} \right) = 0 & \text{if } \{j_1, \dots, j_d\} \notin \mathfrak{B} \end{array} \right. \right\}$$

where G_M^j is the j -th column of G_M and \mathfrak{B} denotes the set of bases of M .

Note A.1. The defining equalities and inequalities of X_M have integer coefficients.

Algorithm 2 FillMatrix

Require: $\text{case} = (d, m)$, a pair from Equation (**); **basis**, a subset of $\{1, \dots, m\}$ of cardinality d .

Ensure: a matrix G , filled with entries belonging to $\{-1, 0, 1\}$ and ensuring Conditions (C2) and (C3) from Definition 2.2.

```

1: Create a  $d \times m$  matrix  $G$ , and fill it with  $-1$  entries.
2: Set  $\text{non\_basis}$  to be equal to the set  $\{1, \dots, m\} \setminus \text{basis}$ .
3:                                      $\triangleright$  Imposing Condition (C2).
4: Insert in  $G$  a  $d \times d$  identity matrix in correspondence to the columns of  $\text{basis}$ .
5:                                      $\triangleright$  Inserting as many zeroes as possible in  $G$ .
6: for  $j$  in  $\text{non\_basis}$  do
7:   for  $i \in \{1, \dots, d\}$  do
8:     if  $(\{1, \dots, d\} \setminus \{i\}) \cup \{j\}$  is not a basis of matroid then
9:       Set  $G(i, j) = 0$ .
10:    end if
11:  end for
12: end for
13:                                      $\triangleright$  Computing the normal frame and imposing Condition (C3).
14:                                      $\triangleright$  Inserting 1s column by column.
15: for  $j$  in  $\text{non\_basis}$  do
16:   Set  $r = 1$ .
17:   while  $G(r, j) = 0$  do
18:     Increase  $r$  by 1.
19:     if  $r = d + 1$  then Break the loop.
20:   end if
21: end while
22:   if  $r \leq d$  then Set  $G(r, j) = 1$ .
23: end if
24: end for
25:                                      $\triangleright$  Inserting 1s row by row.
26: for  $i \in \{1, \dots, d\}$  do
27:   Set  $c = 1$ .
28:   while  $G(i, c) = 0$  or  $G(i, c) = 1$  do
29:     Increase  $c$  by 1.
30:     if  $c = m + 1$  then Break the loop.
31:   end if
32: end while
33:   if  $c \leq m$  then Set  $G(i, c) = 1$ .
34: end if
35: end for
36: return  $G$ .

```

If we compare Definition A.1 and Definition 2.2 it is not hard to see that the quasi-projective variety X_M is isomorphic to the space $\mathcal{R}_{\mathbb{C}}^R(M)$ endowed with the Zariski topology (see Note 2.1 for more details).

Taking this into account, from now on we will be concerned with the determination of the irreducibility of X_M . Notice that if $d = 1$, $d = m$, or $d = m - 1$

the reduced variety X_M is either empty (in which case $\mathcal{R}_{\mathbb{C}}^R(M) = \emptyset$, and so by Proposition 2.1 the matroid M is non-realizable over \mathbb{C}), or equals a point (thus in particular $\mathcal{R}_{\mathbb{C}}^R(M)$ is irreducible).

Hence we are left with the cases when (d, m) belongs to

$$(**) \quad \{(2, 4), (2, 5), (2, 6), (2, 7), (3, 5), (3, 6), (3, 7), (4, 6), (4, 7), (5, 7)\}$$

All matroids in these cases are classified (see [MMIB11]) and the tables describing them are available at

<http://www-imai.is.s.u-tokyo.ac.jp/~ymatsu/matroid/index.html>

For all matroids M in the cases covered by Equation (**) we computed the equalities and inequalities defining X_M . Notice that, if \hat{X}_M is the subset of $\mathbb{A}_{\mathbb{C}}^{s_M}$ defined by

$$\hat{X}_M = \left\{ (z_1, \dots, z_{s_M}) \in \mathbb{A}_{\mathbb{C}}^{s_M} \mid \det \left(G_M^{j_1} | \dots | G_M^{j_d} \right) = 0 \text{ if } \{j_1, \dots, j_d\} \notin \mathfrak{B} \right\}$$

where G_M^j is the j -th column of G_M and \mathfrak{B} is the set of bases of M , then by elementary topology arguments the irreducibility of \hat{X}_M implies the one of X_M , if the latter is non-empty. We checked by that \hat{X}_M is always irreducible, hence we conclude that X_M is always either empty, or irreducible. There are algorithms that decide whether an algebraic set defined by rational equalities (as \hat{X}_M , recall Note A.1) is irreducible or not (see for example [CG05]); in our case, via a direct inspection helped by computations with Sage, we noticed that all sets \hat{X}_M fall into one of these families:

- linear varieties;
- rational hypersurfaces;
- quadrics of rank strictly bigger than 2;

or are cones over such varieties, and so are irreducible by easy algebraic geometry arguments.

The Sage code we used to perform the test is available at the following links:

- <http://perso.unifr.ch/elia.saini/hyperplanes.sage>
- <http://matteogallet.altervista.org/main/papers/hyperplanes2015/hyperplanes.sage>

The Algorithm **TestIrreducibility** provided in 1 describes the pseudocode of the main procedure we implemented and the Algorithm **FillMatrix** presented in 2 sketches the pseudocode of the ancillary algorithm we used to build the matrix G_M (compare the definition of the operations (S1), (S2), (S3), (S4) and (S5)).

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